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Compact Finite Difference Schemes for Partial integro-differential Equations

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Abstract In the present paper a numerical method based on fourth order finite difference and collocation method is presented for the numerical solution of partial integro-differential equation (PIDE). A composite weighted trapezoidal rule is manipulated to handle the numerical integrations which results in a closed-form difference scheme. The efficiency and accuracy of the scheme is validated by its application to one test problem which have exact solutions. Numerical results show that this fourth-order scheme has the expected accuracy. The most advantages of compact finite difference method for PIDE are that it obtains high order of accuracy, while the time complexity to solve the matrix equations after we use compact finite difference method on PIDE is O(*N*), and it can solve very general case of PIDE.

Keywords: Compact finite difference method; PIDE; Partial integro-differential equations; High accuracy; Collocation method.

1 Introduction

Partial integro-differential equation is an equation that the unknown function appears under the sign of integration and contains the derivatives of the unknown function. It can be classified into Fredholm equations and Volterra equations. The upper bound of the region for integral part of Volterra type is variable, while it is a fixed number for that of Fredholm type. In this paper we focus on Volterra integro-differential equation. The fundamental problems on linear second order partial differential equations of parabolic type with different boundary conditions have been substantially investigated in [1-9] and others. Recently, Bange [10] and Pau [11] have reported certain results on the existence and uniqueness of solutions of quasilinear parabolic partial differential equations of second order. The problem of the existence and uniqueness of solution for systems governed by linear integro-partial differential equations of parabolic has been considered in [12].

Consider the following initial boundary value problem for the one-dimensional partial integro-differential equation with memory term,

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=\int_0^t k(t,s)u(x,s)ds+f(x,t), \qquad (1.1)$$
$$x\in[0,1],\ t\in I=[0,T],$$

$$u\big|_{x=a}=g_1(t),\ u\big|_{x=b}=g_2(t),\ t\geq 0, \qquad (1.2)$$

$$u\big|_{t=0}=u_0(x), \quad a\leq x\leq b. \qquad (1.3)$$

Many authors have considered numerical methods for the solution of the linear problem of the form (1.1). Typically, the time discretization is affected by a combination of finite difference and quadratures. Finite difference in time and finite elements in space have been discussed in the case of a smooth kernel (see e.g., Sloan & Thomèe, 1986; Cannon & Lin, 1988, 1990; Yanik & Fairweather, 1988; Thomée & Zhang, 1989; Lin et al., 1991; Zhang, 1993). For the non-smooth kernel case we refer to Chen et al. (1992) and Larsson et al. (1998).

Our contribution in this paper is to develop a new fourth-order accurate scheme for solving partial integro-differential equations in one dimensional space with non-homogeneous Dirichlet boundary conditions. The suggested numerical sch starts with the discretization in time by the 2-point Euler backward finite difference method. After that we deal with a combination of the compact finite difference method and the trapezoidal rule for calculating the integral term and then we use a collocation method to compute the unknown function and finally the obtained system of algebraic equations is solved by iterative methods. The proposed technique is programmed using ***Matlab ver. 7.8.0.347 (R2009a).***

The paper is organized as follows: In Section 2, we give a brief introduction to a high accurate compact finite difference formula for ordinary differential equations and partial integro-differential equations with varying boundary conditions. In Section 3, the proposed scheme is directly applicable to solve one numerical example to support the efficiency of the suggested numerical scheme. Conclusions are drawn in Section 4.

2 Formulations of High-Order Compact Schemes

Compact Schemes are based on a fourth-order accurate approximation to the derivative calculated from ordinary differential equation. To developed the scheme for one-dimensional uniform Cartesian grids with spacing $\Delta x = h$, let us introduce the following notations [13]: If $u_j \equiv u(x_j)$, then we use notations

$$\begin{aligned} \delta_+ u_j &= \frac{u_{j+1} - u_j}{h} = \delta_{x_+} \\ \delta_- u_j &= \frac{u_j - u_{j-1}}{h} = \delta_{x_-} \end{aligned} \tag{2.1}$$

to denote the standard forward finite difference and backward finite difference schemes for first derivative. Also

$$\delta_0\, u_j = \tfrac{1}{2}\left(\delta_+\, u_j + \delta_-\, u_j\right) = \frac{u_{j+1} - u_{j-1}}{2h} \tag{2.2}$$

is the first-order central finite difference with respect to x. The standard second-order central finite difference is denoted as $\delta_x^2 u_j$ and is defined as

$$\delta_+ \delta_- u_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = \frac{\delta_+ - \delta_-}{h}. \tag{2.3}$$

By using the Taylor's series expansion, a fourth orders accurate finite difference for the first and second derivatives can be approximated by

$$\begin{aligned} \delta_0\, u &= \frac{du}{dx} + \frac{h^2}{3!}\frac{d^3u}{dx^3} \\ &= \left(1 + \frac{h^2}{6}\frac{d^2}{dx^2}\right)\frac{du}{dx} = \left(1 + \frac{h^2}{6}\delta^2\right)\frac{du}{dx} + O(h^4) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \delta_x^2\, u &= \frac{d^2u}{dx^2} + \frac{h^2}{12}\frac{d^4u}{dx^4} \\ &= \left(1 + \frac{h^2}{12}\frac{d^2}{dx^2}\right)\frac{d^2u}{dx^2} = \left(1 + \frac{h^2}{12}\delta^2\right)\delta^2 u + O(h^4) \end{aligned} \tag{2.5}$$

2.1 Compact finite difference method for solving ordinary differential equations

In this section, the fourth order compact finite difference method is used to obtain a numerical solution to the following second order ordinary differential equation

$$u''(x) = f(x), \qquad a \le x \le b, \tag{2.1.1}$$

$$u(a) = \beta_1, \qquad u'(a) = \beta_2, \tag{2.1.2}$$

where β_1 and β_2 are constant values. A fourth-order accurate finite difference estimate for $u''(x)$ is,

$$\begin{aligned} \delta_x^2 u_j &= u''_j + \frac{h^2}{12}u_j^{(4)} \\ &= \left(1 + \frac{h^2}{12}\delta^2\right)\delta^2 u + O(h^4) \end{aligned} \tag{2.1.3}$$

Noting that $O(h^2)$ term is included in equation (2.1.3), because we want to approximate it in order to construct an $O(h^4)$ scheme. Applying δ_x^2 to u''_j, we get

$$u_j^{(4)} = \delta_x^2\, u''_j + O(h^2). \tag{2.1.4}$$

Substituting equation (2.1.4) into (2.1.3) yields

$$\delta_x^2 u_j = u''_j + \frac{h^2}{12}\left(\delta_x^2 u''_j + O(h^2)\right) + O(h^4). \tag{2.1.5}$$

From (2.1.1) into (2.1.5)

$$\delta_x^2 u_j = f_j + \frac{h^2}{12}\delta_x^2\ f_j + O(h^4), \tag{2.1.6}$$

Suppose that w_j is the discrete approximation to $u(x_j)$, and using the above scheme we get

$$\begin{aligned} w_{j+1} - 2w_j + w_{j-1} &= \\ &= \frac{h^2}{12}\left(f_{j+1} + 10 f_j + f_{j-1}\right), \end{aligned} \tag{2.1.7}$$

where $f_{j-1}, f_j, f_{j+1}, w_{j-1}$ are known and w_j can be determined from equation (2.1.2), so we can calculate w_{j+1}.

2.2 Compact finite difference method for solving partial integro-differential equations

Here, we use the fourth order compact finite difference method to solve problem (1.1)-(1.3). To construct a numerical solution, we first consider the nodal points (x_j, t_i) defined in the region $[a,b]\times[0,T]$ where

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b, \quad x_{j+1} - x_j = h,$$

and

$$0 = t_0 < t_1 < \cdots < t_i < \cdots < T, \qquad t_{i+1} - t_i = \tau.$$

In such a case we have $x_j = a + jh$ for $j = 0, 1, 2, \ldots, n,$ and $t_i = i\,\tau$ for $i = 0,1,2,\ldots$.

The initial condition in equation (1.2) is approximated as follows:

$$u(x,0) = u_0 = u(x,t_0), \quad a \le x \le b. \qquad (2.2.1)$$

Next, the 2-point Euler backward differentiation formula is manipulated to approximate u_t, given in equation (1.1), at the time-level t_{i+1} for $i = 0, 1, 2, \ldots$. Therefore, we have

$$\frac{u_{i+1}(x) - u_i(x)}{\tau} - \frac{d^2 u_{i+1}(x)}{dx^2} = $$

$$= \int_0^{t_{i+1}} k_{i+1}(s)\, u(x,s)\, ds + f_{i+1}(x), \qquad (2.2.2)$$

where $f_{i+1}(x) = f(x, t_{i+1}), \quad k_{i+1}(s) = k(t_{i+1}, s)$ and $u_{i+1}(x) = u(x, t_{i+1})$. Equivalently, we can rewrite equation (2.2.2) as

$$u''_{i+1}(x) = \frac{u_{i+1}(x) - u_i(x)}{\tau} -$$

$$-\int_0^{t_{i+1}} k_{i+1}(s)\, u(x,s)\, ds - f_{i+1}(x), \qquad (2.2.3)$$

Putting $x = x_j,\ j = 1, \cdots, n-1$ in (2.2.3), then

$$u''_{i+1,\,j} = \frac{u_{i+1,\,j} - u_{i,\,j}}{\tau} -$$

$$-\int_0^{t_{i+1}} k_{i+1}(s)\, u(x_j, s)\, ds - f_{i+1,\,j}, \qquad (2.2.4)$$

where

$u''_{i+1,\,j} = u''(x_j, t_{i+1}),\ u_{i+1,\,j} = u(x_j, t_{i+1}),\ u_{i,\,j} = u(x_j, t_i),$ and $f_{i+1,\,j} = f(x_j, t_{i+1})$. The fourth order accurate finite difference estimate for $u''(x)$ is used from (2.1.5) to give

$$\delta_x^2 u_{i+1,\,j} = u''_{i+1,\,j} + \left(\frac{h^2}{12}\delta_x^2\right)\left(u''_{i+1,\,j}\right) + O(h^4). \qquad (2.2.5)$$

Then, a compact (implicit) approximation for $u''(x)$ with fourth-order accuracy will be given as

$$u''_{i+1,\,j} = \frac{\delta_x^2 u_{i+1,\,j}}{\left(1 + \frac{h^2}{12}\delta_x^2\right)} + O(h^4). \qquad (2.2.6)$$

Using this estimate and considering the discrete solution of equation (2.2.4) which satisfies the approximation, we get

$$\left[1 - \frac{h^2}{12\,\tau}\right]\delta_x^2 u_{i+1,\,j} - \frac{u_{i+1,\,j}}{\tau} + \int_0^{t_{i+1}} k_{i+1}(s)\, u(x_j, s)\, ds +$$

$$+\frac{h^2}{12}\int_0^{t_{i+1}} k_{i+1}(s)\, \delta_x^2 u(x_j, s)\, ds = -\frac{u_{i,\,j}}{\tau} -$$

$$-\frac{h^2}{12\,\tau}\delta_x^2 u_{i,\,j} - f_{i+1,\,j} - \frac{h^2}{12}\delta_x^2 f_{i+1,\,j} \qquad (2.2.7)$$

$$\left[\frac{1}{h^2} - \frac{1}{12\tau}\right]\left(u_{i+1,\,j+1} + u_{i+1,\,j-1}\right) + \left[\frac{-2}{h^2} - \frac{5}{6\tau}\right]u_{i+1,\,j} +$$

$$+\frac{5}{6}\int_0^{t_{i+1}} k_{i+1}(s) u_j(s) ds + \frac{1}{12}\int_0^{t_{i+1}} k_{i+1}(s) u_{j+1}(s) ds$$

$$+\frac{1}{12}\int_0^{t_{i+1}} k_{i+1}(s) u_{j-1}(s) ds = \frac{-1}{12\tau}(u_{i,\,j+1} + u_{i,\,j-1}) -$$

$$-\frac{5}{6\tau}u_{i,\,j} - \frac{5}{6}(f_{i+1,\,j+1} + f_{i+1,\,j-1}) - \frac{1}{12} f_{i+1,\,j}. \qquad (2.2.8)$$

The latest integral will be handled numerically using the composite weighted trapezoidal rule given by:

$$\int_{t_0}^{t_{i+1}} f(s)\, ds \approx \tau \sum_{m=0}^{i}[w\, f(t_m) + (1-w)\, f(t_{m+1})]$$

$$= \tau\left[w f(t_0) + (1-w) f(t_{i+1}) + \sum_{m=1}^{i} f(t_m)\right]. \qquad (2.2.9)$$

Using (2.2.9) we get

$$\int_0^{t_{i+1}} k_{i+1}(s)\, u(x,s)\, ds \approx$$

$$\approx \tau w k_{i+1}(0) u_0(x) + \tau(1-w) k_{i+1}(t_{i+1}) u_{i+1}(x) +$$

$$+\tau\sum_{m=1}^{i}k_{i+1}(t_m)u_{i+1-m}(x) \qquad (2.2.10)$$

The substitutions of this equation into equation (2.2.8) yields

$$\left[\frac{1}{h^2}-\frac{1}{12\tau}\right]\left(u_{i+1,j+1}+u_{i+1,j-1}\right)+\left[\frac{-2}{h^2}-\frac{5}{6\tau}\right]u_{i+1,j}+$$
$$+\frac{5\tau}{6}\left[wk_{i+1}(0)u_{0,j}+(1-w)k_{i+1}(t_{i+1})u_{i+1,j}\right]+$$
$$+\frac{5\tau}{6}\sum_{m=1}^{i}k_{i+1}(t_m)u_{i+1-m,j}+\frac{\tau}{12}wk_{i+1}(0)u_{0,j+1}+$$
$$\frac{\tau}{12}\left[(1-w)k_{i+1}(t_{i+1})u_{i+1,j+1}+\sum_{m=1}^{i}k_{i+1}(t_m)u_{i+1-m,j+1}\right]+$$
$$+\frac{\tau}{12}\left[wk_{i+1}(0)u_{0,j-1}+(1-w)k_{i+1}(t_{i+1})u_{i+1,j-1}\right]+$$
$$+\frac{\tau}{12}\sum_{m=1}^{i}k_{i+1}(t_m)u_{i+1-m,j-1}=\frac{-1}{12\tau}(u_{i,j+1}+u_{i,j-1})-$$
$$-\frac{5u_{i,j}}{6\tau}-\frac{5}{6}(f_{i+1,j+1}+f_{i+1,j-1})-\frac{f_{i+1,j}}{12}. \qquad (2.2.11)$$

Let $U_i(x)$ be a function that approximates $u(x,t_i)$ for the time-level $t_i=i\,\tau$, and is a linear combination of n+1 shape functions which is expressed as:

$$U_i(x)=\sum_{m=0}^{n}c_{mi}\,\varphi_m(x), \qquad (2.2.12)$$

where $\{c_{mi}\}_{m=0}^{n}$ are the unknown real coefficients, to be evaluated, and the $\varphi_m(x)$ are any knowing basis functions

The approximate solutions $u_i(x)$ for different time-levels are determined iteratively as follows. Starting with the time-level $t_0=0$, the value of $u_0(x_j)$, $u_0(x_{j+1})$, and $u_0(x_{j-1})$, for $j=1,2,\ldots,n-1$ are found from equation (1.2). Next, we will approximate the solution u_{i+1} for i = 0 in equation (2.2.8) by the shape functions U_1, as is given in equation (2.2.12). Hence equation (2.2.8) is approximated by:

$$\left(\frac{1}{h^2}-\frac{1}{12\tau}\right)\left(U_{1,j+1}+U_{1,j-1}\right)+\left(\frac{-2}{h^2}-\frac{5}{6\tau}\right)U_{1,j}+$$
$$\frac{5\tau}{6}\left[wk_1(0)u_{0,j}+(1-w)k_1(t_1)U_{1,j}\right]+$$
$$+\frac{\tau}{12}\left[wk_1(0)u_{0,j+1}+(1-w)k_1(t_1)U_{1,j+1}\right]+$$
$$+\frac{\tau}{12}\left[wk_1(0)u_{0,j-1}+(1-w)k_1(t_1)U_{1,j-1}\right]=-\frac{f_{1,j}}{12}$$
$$=\frac{-1}{12\tau}\left[u_{0,j+1}+u_{0,j-1}\right]-\frac{5u_{0,j}}{6\tau}-\frac{5}{6}\left[f_{1,j+1}+f_{1,j-1}\right] \qquad (2.2.13)$$

Replacing U_1 by the approximate solution given by equation (2.2.12) yields the following linear system of $n-1$ equations

$$\left[\frac{1}{h^2}-\frac{1}{12\tau}\right]\left(\sum_{m=0}^{n}c_{m1}\,\phi_{m\;j+1}+\sum_{m=0}^{n}c_{m1}\,\phi_{m\;j-1}\right)+$$
$$+\left[\frac{-2}{h^2}-\frac{5}{6\tau}\right]\sum_{m=0}^{n}c_{m1}\,\phi_{m\;j}+\frac{5\tau(1-w)k_1(t_1)}{6}\sum_{m=0}^{n}c_{m1}\phi_{m\;j}+$$
$$+\frac{\tau(1-w)k_1(t_1)}{12}\left(\sum_{m=0}^{n}c_{m1}\phi_{m\;j+1}+\sum_{m=0}^{n}c_{m1}\phi_{m\;j-1}\right)=$$
$$=\frac{-5}{6}\left[\tau wk_1(0)+\frac{1}{\tau}\right]u_{0,j}-\frac{5}{6}\left(f_{1,j+1}+f_{1,j-1}\right)-$$
$$-\left[\frac{\tau wk_1(0)}{12}+\frac{1}{12\tau}\right](u_{0,j+1}+u_{0,j-1})-\frac{f_{1,j}}{12}, \qquad (2.2.14)$$

where $\sum_{m=0}^{n}c_{m1}\varphi_{m\;j+1}=\sum_{m=0}^{n}c_{m1}\varphi_m(x_{j+1}),$. Rewrite equation (2.1.17) as

$$\sum_{m=0}^{n}c_{m1}\left[a_1\varphi_{m\;j+1}+a_2\varphi_{m\;j}+a_1\varphi_{m\;j-1}\right]=$$
$$=a_3\,u_{0,j}+a_4\,(u_{0,j+1}+u_{0,j-1})-\frac{1}{12}\,f_{1,j}-$$
$$-\frac{5}{6}(f_{1,j+1}+f_{1,j-1}), \qquad (2.2.15)$$

where

$$\begin{cases} a_1=\dfrac{1}{h^2}-\dfrac{1}{12\,\tau}+\dfrac{\tau}{12}(1-w)\,k_1(t_1)\\ a_2=\dfrac{-2}{h^2}-\dfrac{5}{6\,\tau}+\dfrac{5\,\tau}{6}(1-w)\,k_1(t_1)\\ a_3=\dfrac{-5}{6}\left(\tau\,w\,k_1(0)+\dfrac{1}{\tau}\right)\\ a_4=-\left[\dfrac{\tau\,w\,k_1(0)}{12}+\dfrac{1}{12\,\tau}\right]\end{cases} \qquad (2.2.16)$$

The system (2.2.15) consists of $(n+1)$ equation in the $(n+1)$ unknowns $\{c_{m1}\}_{m=0}^{n}$. To get a solution of this system we need two additional conditions. These conditions are obtained from the boundary conditions (1.2)

$$\sum_{m=0}^{n} c_{m1}\varphi_m(a) = g_1(t_i),\ i = 0,\dots,n \quad (2.2.17)$$

$$\sum_{m=0}^{n} c_{m1}\phi_m(b) = g_2(t_i),\ i = 0,\dots,n \quad (2.2.18)$$

Since f and u_0 are known at every grid point, the right hand side of equation (2.2.15) is known for all nodes. The system (2.2.15), equations (2.2.17) and (2.2.18) consist of $(n+1)$ equations in $(n+1)$ unknowns; this system is of the form

$$AC = F. \quad (2.2.19)$$

Upon solving the system (2.2.19), the function $u_1(x)$ is approximated by the sum:

$$u_1(x_j) = \sum_{m=0}^{n} c_{m1}\phi_m(x_j),\ j = 0,1,2,\dots,n. \quad (2.2.20)$$

Next, we find the approximate solution at time-levels $t_2, t_3, \dots$ recursively by solving the following system for $i = 2, 3, \dots$.

$$\sum_{m=0}^{n} c_{mi}\left(a_1\phi_{m\ j+1} + a_2\phi_{m\ j} + a_1\phi_{m\ j-1}\right) =$$

$$= \frac{-1}{12\tau}\left(u_{i-1,\,j+1} + u_{i-1,\,j-1}\right) - \frac{5}{6\tau}u_{i-1,\,j} -$$

$$-\frac{1}{12\tau}(u_{0,\,j+1} + u_{0,\,j-1}) - \frac{5}{6}\left(f_{i,\,j+1} + f_{i,\,j-1}\right) -$$

$$-\frac{1}{12}f_{i,\,j} - \frac{5}{6\tau}u_{0,\,j} - \frac{5\tau}{6}\sum_{m=1}^{i} k_i(t_m)u_{i-m,\,j} -$$

$$\frac{\tau}{12}\sum_{m=1}^{i} k_i(t_m)\left(u_{i-m,\,j+1} - u_{i-m,\,j-1}\right), \quad (2.2.21)$$

$$\sum_{m=0}^{n} c_{m1}\varphi_m(a) = g_1(t_i),\ i = 0,\dots,n \quad (2.2.22)$$

$$\sum_{m=0}^{n} c_{m1}\phi_m(b) = g_2(t_i),\ i = 0,\dots,n \quad (2.2.23)$$

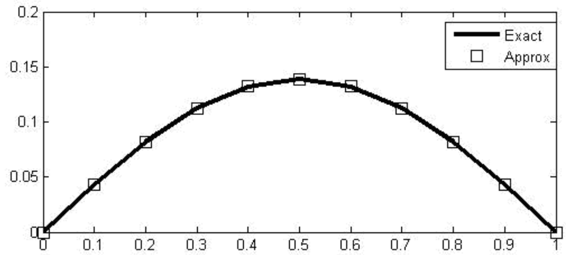
3 Numerical Experiment

In this section, we solve the integro-differential equation (1.1)-(1.3) in $(0,1)\times(0,T)$ with

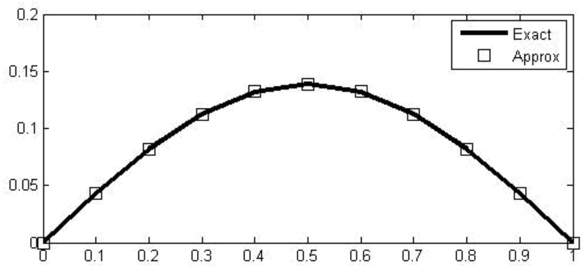
$k(s,t) = s\,t,\ \ f(t,x) = \sin(\pi x)(-t\,\frac{e^{-\pi^2 t}}{\pi^2} - \frac{e^{-\pi^2 t}}{\pi^4} + \frac{1}{\pi^4})$ $u_0 = \sin(\pi x)$, and $g_1(t) = g_2(t) = 0$. The theoretical solution of this problem is $u(t,x) = e^{-\pi^2 t}\sin(\pi x)$.

We employ a compact difference scheme for the space derivative so that we get a full discretization scheme with error estimation $\mathrm{O}(h^4) + \mathrm{O}(\tau)$. We shall compare the results obtained by the suggested approximation scheme with the exact solution.

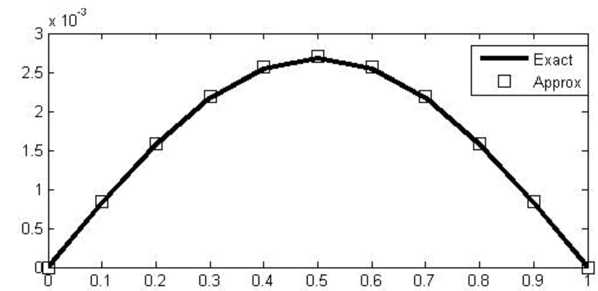
Behavior of numerical and exact at t=0.2 and time step= 0.00001



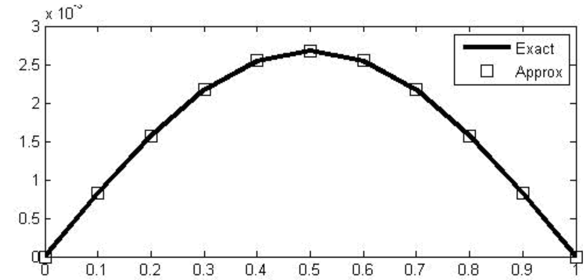
Behavior of numerical and exact at t=0.2 and time step= 0.001



Behavior of numerical and exact at t=0.6 and time step= 0.001



Behavior of numerical and exact at t=0.6 and time step= 0.00001



Behavior of numerical and exact at t=0.8 and time step= 0.0001

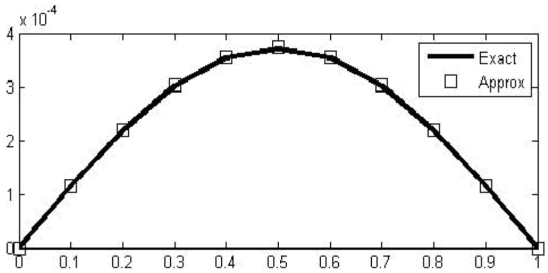
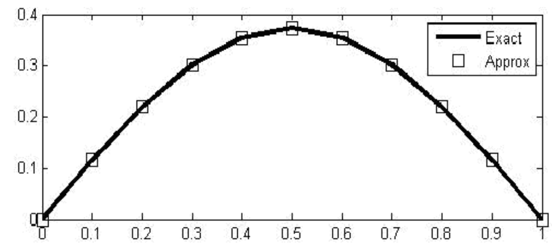


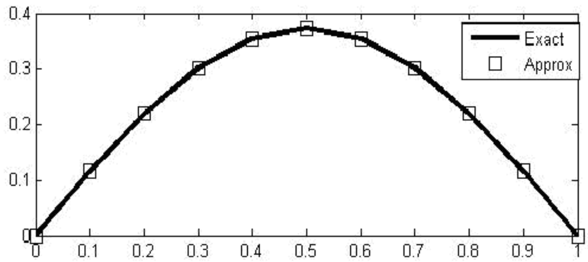
Figure (1), the behavior of numerical and exact solution at different values of time.

Figure (2), the behavior of numerical and exact solution at different values of time.

Behavior of numerical and exact at t=0.1 and time step= 0.001



Behavior of numerical and exact at t=0.1 and time step= 0.0001



Behavior of numerical and exact at t=0.8 and time step= 0.001

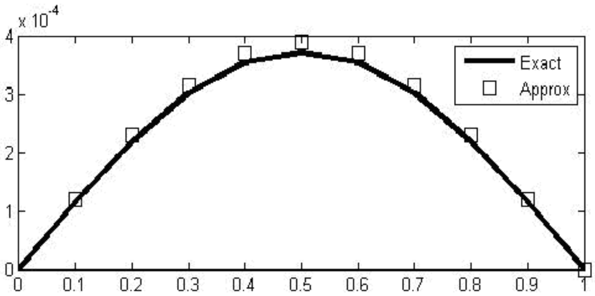


Table 1.
Comparison between exact and numerical solutions

x	t = 0.2, n = 10		
	Exact solution	Present method	
		τ = 0.001	τ = 0.00001
0.0	0	0	0
0.1	4.2925E-002	4.2931E-002	4.2925E-002
0.2	8.1649E-002	8.1659E-002	8.1649E-002
0.3	1.1238E-001	1.1239E-001	1.1238E-001
0.4	1.3211E-001	1.3212E-001	1.3211E-001
0.5	1.3891E-001	1.3892E-001	1.3891E-001
0.6	1.3211E-001	1.3212E-001	1.3211E-001
0.7	1.1238E-001	1.1239E-001	1.1238E-001
0.8	8.1649E-002	8.1659E-002	8.1649E-002
0.9	4.2925E-002	4.2931E-002	4.2925E-002
1.0	0	0	0

Table 2.
Comparison between exact and numerical solutions

x	t = 0.6, n = 20		
	Exact solution	Present method	
		τ = 0.001	τ = 0.00001
0.0	0	0	0
0.1	8.2831E-004	8.3369E-004	8.2836E-004
0.2	1.5755E-003	1.5857E-003	1.5756E-003
0.3	2.1685E-003	2.1826E-003	2.1686E-003
0.4	2.5492E-003	2.5658E-003	2.5494E-003
0.5	2.6804E-003	2.6978E-003	2.6806E-003
0.6	2.5492E-003	2.5658E-003	2.5494E-003
0.7	2.1685E-003	2.1826E-003	2.1686E-003
0.8	1.5755E-003	1.5857E-003	1.5756E-003
0.9	8.2831E-004	8.3369E-004	8.2836E-004
1.0	0	0	0

Table 3.
Comparison between exact and numerical solutions

x	$t = 0.1$, $n = 20$		
	Exact solution	Present method	
		$\tau = 0.001$	$\tau = 0.0001$
0.0	0	0	0
0.1	1.1517E-001	1.1518E-001	1.1517E-001
0.2	2.1907E-001	2.1908E-001	2.1907E-001
0.3	3.0152E-001	3.0154E-001	3.0152E-001
0.4	3.5446E-001	3.5448E-001	3.5446E-001
0.5	3.7270E-001	3.7273E-001	3.7270E-001
0.6	3.5446E-001	3.5448E-001	3.5446E-001
0.7	3.0152E-001	3.0154E-001	3.0152E-001
0.8	2.1907E-001	2.1908E-001	2.1907E-001
0.9	1.1517E-001	1.1518E-001	1.1517E-001
1.0	0	0	0

Table 4.
Comparison between exact and numerical solutions

x	$t = 0.8$, $n = 20$		
	Exact solution	Present method	
		$\tau = 0.001$	$\tau = 0.0001$
0.0	0	0	0
0.1	1.1506E-004	1.2049E-004	1.1560E-004
0.2	2.1886E-004	2.2919E-004	2.1990E-004
0.3	3.0123E-004	3.1545E-004	3.0266E-004
0.4	3.5412E-004	3.7084E-004	3.5580E-004
0.5	3.7234E-004	3.8992E-004	3.7411E-004
0.6	3.5412E-004	3.7084E-004	3.5580E-004
0.7	3.0123E-004	3.1545E-004	3.0266E-004
0.8	2.1886E-004	2.2919E-004	2.1990E-004
0.9	1.1506E-004	1.2049E-004	1.1560E-004
1.0	0	0	0

4 Conclusions

A fourth-order accurate compact finite difference scheme for partial integro-differential problems was developed. The method reduces the underlying problem to linear system of algebraic equations, which can be solved successively to obtain a numerical solution at varied time-levels. Numerical experiments which shown in the above scheme are good agreement with the exact ones. Moreover, the results in tables (1-4) and figures (1, 2) confirm that the numerical solutions can be refined when the time-step τ is reduced, or the number of nodes is increased.

References

[1] Aronson, D. G., Non-negative solutions of linear parabolic equations. Ann. Scuola Norm. Sup. Pisa Cl. Fis. Mat. 1 22 (1968), 607-694.

[2] Aronson, D. G. and Serrin, James, Local behaviour of solutions of quasilinear parabolic equations. Arch. Rational Mech. Anal. 25 (1967), 81-122.

[3] Eklund, N. A., Existence and uniqueness of solutions of parabolic equations. Proc. Amer. Math. Soc. 47 (1975), 137-142.

[4] Eklund, N. A., Boundary behaviour of solutions of parabolic equations with discontinuous coefficients. Bull. Amer. Math. Society 77 (1971), 788-792.

[5] Fleming, W., The Cauchy problem for degenerate parabolic equations. J. Math. Mech. 13 (1964). 987-1008.

[6] Friedman, A., Partial differential equations of parabolic type. Prentice-Hall, Englewood Cliffs, N. J., 1964.

[7] Il'in, A. M., Kalashnikov, A. S. and Oleinik, O. A., Second order linear equations of parabolic type. Russian Math. Surveys 17 (1962), 1-143.

[8] Ladyzenskaya, O. A. and Ural'ceva, M. N., Boundary problems for linear and quasilinear parabolic equations. In Amer. Math. Soc. Transl. 2 No. 47, Amer. Math. Soc., Providence, R. I., 1965, pp. 217-299.

[9] Ladyzenskaya, O. A., Solonnikov, V. A. and Ural'ceva, N. N., linear and quasilinear equations of parabolic type. Amer. Math. Soc., Providence, R. I., 1968.

[10] Bange, D. W., A constructive existence theorem for a nonlinear parabolic equation. SIAM I. Math. Anal. 5 (1974), 91-102.

[11] Pao, C. V., Successive approximations of some nonlinear-boundary value problems. Siam j. Math. Anal. 5 (1974), 91-102.

[12] I. H. Sloan and V. Thomèe, Time discretization of an integro-differential equation of parabolic type, SIAM J. Numer. Anal., 23(1986), pp. 1052-1061.

[13] J.C. Strikwerda, Finite Difference Schemes and Partial Differential Equations, SIAM, Philadelphia, 2nd . Ed., 2004

[14] K. Atkinson, The Numerical Solution of Integral Equations of the Second Kind, Cambridge University press, (1997).